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The Lie group of Newton's and Lagrange's equations for the harmonic oscillator†

C E Wulfman and B G Wybourne‡

Department of Physics, University of the Pacific, Stockton, California 95204, USA

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Abstract. Lie's theory of differential equations is applied to the equation of motion of the classical one-dimensional harmonic oscillator. The equation is found to be invariant under a global Lie group of point transformations that is shown to be SL(3, R). The physical significance of the analysis and the results is considered. It is shown that the periodicity of the motion is a local topological property of the equation, while the length of the periods depends upon global properties.

1. Introduction

The determination of the group of transformations that leaves invariant a given differential equation derives from the early work of Lie (cf Lie 1891, 1922). Though Lie's discoveries have stimulated many developments in mathematics and physics, for a variety of historical reasons his contributions to algebra have received more attention than his contributions to the analysis of differential equations. Aside from the development of the abstract theory of topological groups, this later aspect remained for half a century in much the state that Lie left it at the time of his death at the turn of the century. Then in the nineteen fifties and sixties Lie's approach was extended to partial differential equations of arbitrary order (Ovsjannikov 1962) and to boundary value problems (Bluman 1967, Bluman and Cole 1969). Subsequently it was shown that it is important to consider the invariance of partial differential equations under a wider class of continuous transformations than had previously been recognized (Anderson *et al* 1972a, b, c).

In recent years group theoretical analyses of specific differential equations have also been finding increasing application, for example, in studies of heat transport (Bluman and Cole 1969, Bluman 1971), hydrodynamics (Ovsjannikov 1962, Rosen and Ullrich 1973), and chemical kinetics (Wulfman and Shibuya 1973). Many applications to engineering problems are collected in the books of Ames (Ames 1965, 1972). Applications of the theory of continuous groups in quantum mechanics are legion, and not even a listing of recent work can be given here. However we would call the reader's attention to several articles on the quantum mechanical harmonic oscillator (Baker 1956, Bargmann and Moshinsky 1960, 1961, Barut 1965).

† Investigation supported by a grant from the Research Corporation.
‡ New Zealand Fulbright Grantee. Permanent address: Physics Department, University of Canterbury, Christchurch, New Zealand.

Lie himself first obtained the generators of the invariance group of Newton's equation for the free particle. However it appears that the corresponding analysis for the harmonic oscillator was first carried out by R L Anderson (Anderson and Davison 1974). In this paper we sketch Anderson's derivation for completeness, and then classify the algebra and determine the global Lie group of the equation of motion. The algebra is shown to be a non-compact realization of A_2, and the global group is shown to be SL(3, R). It is shown that, simply because the algebra has a compact subalgebra containing the generator of time translations, the motion of the oscillator is periodic. This fact, and the length of the period, are determined without reference to the solutions of the equation of motion. In the remainder of the paper, a number of further consequences of the group structure are discussed.

2. The infinitesimal transformations

With an appropriate choice of units, Newton's or Lagrange's equation of motion for a one-dimensional harmonic oscillator may be written as

$$d^2x/dt^2+x=0, \qquad \text{or} \qquad \ddot{x}+x=0. \tag{1}$$

We seek those infinitesimal transformations of x and t that leave the equation of motion form invariant—hence interconvert its solutions.

Consider a transformation that carries a point (x, t) into a point (x', t') such that

$$x'=\Phi(x, t, a_0+\delta a) \qquad t'=\Psi(x, t, a_0+\delta a), \tag{2}$$

where for the identity transformation $x=\Phi(x, t, a_0)$ and $t=\Psi(x, t, a_0)$. Then the infinitesimal change in x and t due to the infinitesimal variation δa of the parameter a is given by

$$\delta x=\xi\delta a \qquad \delta t=\eta\delta a \tag{3}$$

where

$$\xi=\xi(x, t)=(\partial\Phi/\partial a)_{a_0} \qquad \eta=\eta(x, t)=(\partial\Psi/\partial a)_{a_0}. \tag{4}$$

If $f(x, t)$ is an analytic function of x, t then under an infinitesimal transformation

$$\delta f=Uf\delta a, \tag{5}$$

where

$$U=\xi\partial/\partial x+\eta\partial/\partial t. \tag{6}$$

The differential equation of interest is of second order, so it is necessary to consider the second extension of the point transformation (2). The infinitesimal operator U'' of the second extended transformation is of the form (cf Lie 1891, 1922, Cohen 1931)

$$U''=U+\xi'\partial/\partial\dot{x}+\xi''\partial/\partial\ddot{x} \tag{7}$$

where

$$\xi'=\delta\dot{x}/\delta a=\partial\xi/\partial t+(\partial\xi/\partial x-\partial\eta/\partial t)\dot{x}-(\partial\eta/\partial x)\dot{x}^2 \tag{8}$$

and

$$\begin{aligned}\xi''=\delta\ddot{x}/\delta a=\partial^2\xi/\partial t^2+(2\partial^2\xi/\partial x\partial t-\partial^2\eta/\partial t^2)\dot{x}+(\partial^2\xi/\partial x^2-2\partial^2\eta/\partial x\partial t)\dot{x}^2-(\partial^2\eta/\partial x^2)\dot{x}^3\\+(\partial\xi/\partial x-2\partial\eta/\partial t-3\dot{x}\partial\eta/\partial x)\ddot{x}.\end{aligned} \tag{9}$$

The equation of motion of the oscillator will be form invariant under the transformations generated by U, U', U'' if and only if

$$U''(\ddot{x}+x)=0 \qquad \text{whenever} \qquad \ddot{x}+x=0, \tag{10}$$

leading to the condition that

$$\xi+\partial^2\xi/\partial t^2-(\partial\xi/\partial x-2\partial\eta/\partial t)x+(2\partial^2\xi/\partial x\partial t-\partial^2\eta/\partial t^2+3x\partial\eta/\partial x)\dot{x}$$
$$+(\partial^2\xi/\partial x^2-2\partial^2\eta/\partial x\partial t)\dot{x}^2-(\partial^2\eta/\partial x^2)\dot{x}^3=0. \tag{11}$$

For equation (11) to hold for all values of the variables, it must be true that

$$\partial^2\eta/\partial x^2=0 \tag{12a}$$
$$\partial^2\xi/\partial x^2-2\partial^2\eta/\partial x\partial t=0 \tag{12b}$$
$$2\partial^2\xi/\partial x\partial t-\partial^2\eta/\partial t^2+3x\partial\eta/\partial x=0 \tag{12c}$$
$$\partial^2\xi/\partial t^2-x\partial\xi/\partial x+2x\partial\eta/\partial t+\xi=0. \tag{12d}$$

The above four equations may be integrated to give

$$U=\sum_{i=1}^{8} b_i X_i \tag{13}$$

where the b_i are integration constants and the X_i are the following operators (or any linearly independent linear combinations of them):

$$\begin{aligned}
X_1&=(1+x^2)\sin t\,\partial/\partial x-x\cos t\,\partial/\partial t\\
X_2&=(1-x^2)\sin t\,\partial/\partial x+x\cos t\,\partial/\partial t\\
X_3&=(1+x^2)\cos t\,\partial/\partial x+x\sin t\,\partial/\partial t\\
X_4&=(1-x^2)\cos t\,\partial/\partial x-x\sin t\,\partial/\partial t\\
X_5&=\partial/\partial t\\
X_6&=x\partial/\partial x\\
X_7&=x\cos 2t\,\partial/\partial x+\sin 2t\,\partial/\partial t\\
X_8&=-x\sin 2t\,\partial/\partial x+\cos 2t\,\partial/\partial t.
\end{aligned} \tag{14a}$$

The coefficients ξ' of the first extensions of these operators are:

$$\begin{aligned}
\xi_1'&=x\dot{x}\sin t+(1+x^2+\dot{x}^2)\cos t\\
\xi_2'&=-x\dot{x}\sin t+(1-x^2-\dot{x}^2)\cos t\\
\xi_3'&=x\dot{x}\cos t-(1+x^2+\dot{x}^2)\sin t\\
\xi_4'&=-x\dot{x}\cos t+(-1+x^2+\dot{x}^2)\sin t\\
\xi_5'&=0\\
\xi_6'&=\dot{x}\\
\xi_7'&=-\dot{x}\cos 2t-2x\sin 2t\\
\xi_8'&=\dot{x}\sin 2t-2x\cos 2t.
\end{aligned} \tag{14b}$$

The corresponding coefficients of the second extensions are:

$$\begin{aligned}
\xi_1'' &= -(1+x^2)\sin t + 3\dot{x}(x+\ddot{x})\cos t \\
\xi_2'' &= -(1-x^2)\sin t - 3\dot{x}(x+\ddot{x})\cos t \\
\xi_3'' &= -(1+x^2)\cos t - 3\dot{x}(x+\ddot{x})\sin t \\
\xi_4'' &= -(1-x^2)\cos t + 3\dot{x}(x+\ddot{x})\sin t \\
\xi_5'' &= 0 \\
\xi_6'' &= \ddot{x} \\
\xi_7'' &= -4x\cos 2t - 3\ddot{x}\cos 2t \\
\xi_8'' &= 4x\sin 2t + 3\ddot{x}\sin 2t.
\end{aligned} \qquad (14c)$$

We have made use of the freedom to pick linear combinations of operators so as to ensure that the X_i of equations (14) are a basis for a Lie algebra having a diagonal metric tensor (see next section).

3. The Lie algebra

The extended operators obey the same commutation rules as the X_i;

$$[X_i, X_j] = c_{ij}^k X_k, \qquad [X_i', X_j'] = c_{ij}^k X_k', \qquad \text{etc} \qquad (15)$$

where the c_{ij}^k are structure constants. The complete set of commutators is given in table 1. The X_i clearly satisfy the requirements of a Lie algebra.

That this Lie algebra is semi-simple may be seen by constructing the metric tensor

$$g_{ij} = c_{ik}^m c_{jm}^k \qquad (16)$$

and showing that the determinant of g_{ij} is non-vanishing, as required by Cartan's criterion for semi-simplicity. Use of table 1 shows that g_{ij} is diagonal with

$$g_{ii} = \begin{cases} +12 & i = 2, 4, 7, 8 \\ +4 & i = 6 \\ -12 & i = 1, 3, 5. \end{cases}$$

Table 1. The commutators $[X_i, X_j]$ of the infinitesimal operators.

X_i	X_j							
	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_1	0	X_7-3X_6	X_5	X_8	$-X_3$	X_2	$-X_2$	$-X_4$
X_2	$-X_7+3X_6$	0	$-X_8$	$-X_5$	$-X_4$	X_1	$-X_1$	$-X_3$
X_3	$-X_5$	X_8	0	$-X_7-3X_6$	X_1	X_4	X_4	$-X_2$
X_4	$-X_8$	X_5	X_7+3X_6	0	X_2	X_3	X_3	$-X_1$
X_5	X_3	X_4	$-X_1$	$-X_2$	0	0	$2X_8$	$-2X_7$
X_6	$-X_2$	$-X_1$	$-X_4$	$-X_3$	0	0	0	0
X_7	X_2	X_1	$-X_4$	$-X_3$	$-2X_8$	0	0	$-2X_5$
X_8	X_4	X_3	X_2	X_1	$2X_7$	0	$2X_5$	0

Since the metric is indefinite we conclude that the Lie algebra is that of a Lie group which is non-compact. The three operators X_1, X_3, X_5 form a compact subalgebra associated with a negative definite metric $g_{ij} = -2\delta_{ij}(i, j = 1, 3, 5)$.

Linear combinations of the X_i may be formed so as to cast the Lie algebra into the Cartan–Weyl standard form, leading to its identification as a non-compact (real) form of Cartan's A_2 algebra. In particular we find

$$\begin{aligned}
&H_1 = (\mathrm{i}/2)\partial/\partial t && H_2 = \tfrac{1}{2}(3)^{1/2} x\partial/\partial x \\
&E_\alpha = \mathrm{e}^{-2\mathrm{i}t}(\mathrm{i}\partial/\partial t + x\partial/\partial x) && E_{-\alpha} = \mathrm{e}^{2\mathrm{i}t}(\mathrm{i}\partial/\partial t - x\partial/\partial x) \\
&E_\beta = \mathrm{e}^{-\mathrm{i}t}(x^2\partial/\partial x + \mathrm{i}x\partial/\partial t) && E_{-\beta} = \mathrm{e}^{\mathrm{i}t}\partial/\partial x \\
&E_\gamma = \mathrm{e}^{-\mathrm{i}t}\partial/\partial x && E_{-\gamma} = \mathrm{e}^{\mathrm{i}t}(x^2\partial/\partial x - \mathrm{i}x\partial/\partial t).
\end{aligned} \tag{17}$$

The roots $\pm\alpha$, $\pm\beta$, $\pm\gamma$ are respectively $(\pm 1, 0)$, $(\pm\frac{1}{2}, \pm\frac{1}{2}(3)^{1/2})$, $(\pm\frac{1}{2}, \mp(3)^{1/2})$.

4. Finite transformations and path curves

Different linear combinations of the generators X_i are the generators of different infinitesimal and finite transformations. For simple operators the effect of a finite transformation may be most easily determined by exponentiation of the infinitesimal operators. The results listed for X_3 and X_4 in table 2 have been obtained in this way. For the remaining operators it is simpler to integrate the system of differential equations

$$\frac{\mathrm{d}x'}{\xi(x', t')} = \frac{\mathrm{d}t'}{\eta(x', t')} = \mathrm{d}a. \tag{18}$$

For future reference we shall consider the process in some detail for the case of X_1. Integrating

$$\frac{\mathrm{d}x'}{(1+x'^2)\sin t'} = \frac{\mathrm{d}t'}{-x'\cos t'} \tag{19}$$

one finds the path curves, or invariant functions, of the one-parameter group generated by X_1. These are

$$(1+x'^2)^{-1}\cos^2 t' = (1+x^2)^{-1}\cos^2 t = k^2; \qquad 0 \leqslant k^2 \leqslant 1. \tag{20}$$

Representative members of this family of curves are sketched in figure 1. Now, to integrate

$$\frac{\mathrm{d}x'}{(1+x'^2)\sin t'} = \mathrm{d}a \tag{21}$$

we use equation (20) to eliminate $\sin t'$, and make the substitution

$$u = x(1+x^2)^{-1/2}, \qquad x = u(1-u^2)^{-1/2} \tag{22}$$

Integration then gives

$$u' = (1-k^2)^{1/2}\sin(\pm a + \alpha); \qquad \alpha = \sin^{-1}[x(1+x^2-\cos^2 t)^{-1/2}]. \tag{23}$$

We may choose the + sign without loss of generality. Transformations with positive values of a then carry the initial point (x, t) counterclockwise along the path curve as a is increased until one reaches the final value of a and the point (x', t'). Negative values of

Table 2. Finite transformations.

Generator	Invariant	$x'(x, t, a)$	$t'(x, t, a)$†	α
X_1	$(1+x^2)^{-1}\cos^2 t$	$\dfrac{(1+x^2-\cos^2 t)^{1/2}\sin(a+\alpha)}{[1+x^2-(1+x^2-\cos^2 t)\sin^2(a+\alpha)]^{1/2}}$	$\cos^{-1}\left[\left(\dfrac{1+x'^2}{1+x^2}\right)^{1/2}\cos t\right]$	$\sin^{-1}[x(1+x^2-\cos^2 t)^{-1/2}]$
X_2	$(1-x^2)^{-1}\cos^2 t$	$\dfrac{(1-x^2-\cos^2 t)^{1/2}\sinh(a+\alpha)}{[1-x^2+(1-x^2-\cos^2 t)\sinh^2(a+\alpha)]^{1/2}}$	$\cos^{-1}\left[\left(\dfrac{1-x'^2}{1-x^2}\right)^{1/2}\cos t\right]$	$\sinh^{-1}[x(1-x^2-\cos^2 t)^{-1/2}]$
X_3	$(1+x^2)^{-1}\sin^2 t$	$\dfrac{(1+x^2-\sin^2 t)\sin(a+\alpha)}{[1+x^2-(1+x^2-\sin^2 t)\sin^2(a+\alpha)]^{1/2}}$	$\sin^{-1}\left[\left(\dfrac{1+x'^2}{1+x^2}\right)^{1/2}\sin t\right]$	$\sin^{-1}[x(1+x^2-\sin^2 t)^{-1/2}]$
X_4	$(1-x^2)^{-1}\sin^2 t$	$\dfrac{(1-x^2-\sin^2 t)\sinh(a+\alpha)}{[1-x^2+(1-x^2-\sin^2 t)\sinh^2(a+\alpha)]^{1/2}}$	$\sin^{-1}\left[\left(\dfrac{1-x'^2}{1-x^2}\right)^{1/2}\sin t\right]$	$\sinh^{-1}[x(1-x^2-\sin^2 t)^{-1/2}]$
X_5	x	x	$t+a$	
X_6	t	$x\,e^a$	t	
X_7	$x^{-4}\sin^2 t$	$\dfrac{x\,e^a \sec t}{(1+e^{4a}\tan^2 t)^{1/2}}$	$\tan^{-1}(e^{2a}\tan t)$	
X_8	$x^{-4}\cos^2 t$	$\dfrac{x\,e^a \operatorname{cosec} t}{(1+e^{4a}\cot t)^{1/2}}$	$\cot^{-1}(e^{2a}\cot t)$	

† cf text, § 4. The branch of each inverse trigonometric function must be chosen to agree with local coordinate systems defined by equations (18).

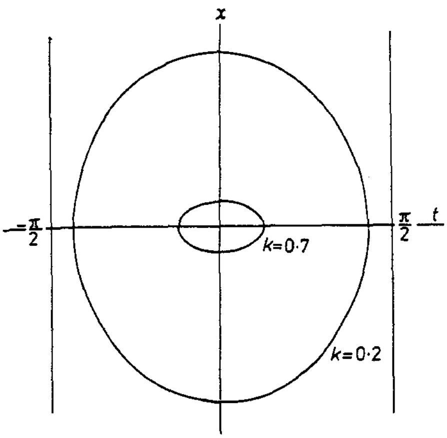

Figure 1.

a correspond to motion in the inverse direction. With this convention,

$$x' = \frac{(1+x^2-\cos^2 t)^{1/2}\sin(a+\alpha)}{[1+x^2-(1+x^2-\cos^2 t)\sin^2(a+\alpha)]^{1/2}}. \tag{24}$$

It follows from this result and equation (20) that the finite transformation of t gives a value of t' on the path curve such that

$$\cos^{-1}\left[\left(\frac{1+x'^2}{1+x^2}\right)^{1/2}\cos t\right] = t', \tag{25a}$$

the branch of the inverse trigonometric function being chosen for which

$$dt'/da = -x'\cos t. \tag{25b}$$

If, for the moment, we eliminate from consideration the exceptional path curves for which $k=0$, and $k=\pm 1$, then it is easy to determine that changing the value of a continuously from $a=0$ to $a=\pi$ carries a point half-way around the path curve, while varying a from $a=0$ to $a=-\pi$ carries it half-way round in the reverse direction. The transformations for which $a=\pm\pi$ give identical results. Thus, for all these unexceptional path curves we may choose the range of a to be

$$-\pi \leqslant a \leqslant \pi; \qquad -\pi \equiv \pi. \tag{26}$$

We now consider the exceptional path curves. When $\xi(x,t)=\eta(x,t)=0$ for some point (x,t), this point will be left invariant by the finite transformation defined by ξ and η. For the transformations generated by X_1 this happens if $x=0$, $\sin t=0$. The invariant points are therefore $(x,t)=(0,n\pi)$, $n=0,\pm 1,\pm 2\ldots$. These points all lie on path curves for which $k^2=1$, curves which have degenerated into single points. As $k\to 0$ the path curves approach a series of straight lines parallel to the x axis and intercepting the t axis at $t=m\pi/2$, $m=\pm 1$, $\pm 3\ldots$. The closed path curves of maximum extent in the t direction are obtained when k differs infinitesimally from zero. It is important to note that the closed curves centred at $t=0$ do not touch those centred at $t=\pi$, etc. Thus in all cases as one varies a over the range (26) an initial point (x,t) is moved continuously through every point on each of the closed path curves, and the motion in the t direction is bounded. When $k=0$ the group generated by X_1

continues to act transitively on the path curve, though the motion in x becomes unbounded.

Equation (26) verifies the conclusion of § 3 that X_1 generates a compact subgroup.

5. The three-parameter compact subgroup: oscillator time

We have seen that the operators X_1, X_3, X_5 generate a compact subgroup. *A priori* this might be either SU(2) or SO(3). In this section we show that the subgroup is in fact SO(3).

Consider an operator of the subgroup of the form

$$G(\boldsymbol{a}) = \exp(a_1X_1 + a_3X_3 + a_5X_5) \tag{27}$$

which acts on the coordinates (x, t). If (x', t') is another pair of coordinates related to (x, t) by the transformation

$$(x', t') = G(\boldsymbol{b})(x, t) \tag{28}$$

then, in the x', t' system, the action of $G(a)$ is given by the conjugate operator

$$G(\boldsymbol{a}') = G(\boldsymbol{b})G(\boldsymbol{a})G(\boldsymbol{b})^{-1}. \tag{29}$$

The transformation $G(\boldsymbol{a}) \leftrightarrow G(\boldsymbol{a}')$ is an inner automorphism of the group, hence a homeomorphism, so the topological properties of $G(\boldsymbol{a}')$ are the same as those of $G(\boldsymbol{a})$ (cf Pontryagin 1966 especially §§§ 17, 24, 41).

The parameters $\boldsymbol{a}'$ are related to the parameters $\boldsymbol{a}$ by the action of the adjoint group

$$(a_1', a_3', a_5') = g(b)(a_1, a_3, a_5) \tag{30}$$

where

$$g(\boldsymbol{b}) = \exp(b_1Y_1 + b_3Y_3 + b_5Y_5). \tag{31}$$

The generators Y_1 of the adjoint group, determined solely by the commutation relations of the X_i, are (cf Racah 1965)

$$\begin{aligned} Y_1 &= -a_3\partial/\partial a_5 + a_5\partial/\partial a_3 \\ Y_3 &= -a_5\partial/\partial a_1 + a_1\partial/\partial a_5 \\ Y_5 &= -a_1\partial/\partial a_3 + a_3\partial/\partial a_1. \end{aligned} \tag{32}$$

It is evident that the adjoint group leaves invariant the quadratic form $a_1^2 + a_3^2 + a_5^2$, and that the group acts transitively on each of its invariant surfaces, $\boldsymbol{a}^2 = \text{constant}$. We may furthermore choose $\boldsymbol{b}$ such that in the new system of coordinates the group action is given entirely by the action of the one-parameter subgroup generated by X_1, i.e., such that $a_3' = a_5' = 0$. In this case we must have

$$a_1'^2 = a_1^2 + a_3^2 + a_5^2 \leqslant \pi^2, \qquad -\pi \equiv \pi. \tag{33}$$

The group in question is therefore SO(3).

We may draw further consequences from the argument of the previous paragraphs if we consider a transformation in which a_1 and a_3 are zero. We then have

$$a_5^2 \leqslant \pi^2; \qquad -\pi \equiv \pi. \tag{34}$$

As X_5 is the time translation generator $\partial/\partial t$, this implies that if $f(t)$ is any solution of the equation of motion then

$$f(t-\pi)=f(t+\pi) \tag{35}$$

for any choice of t. The motion is therefore cyclic with period 2π.

In so far as the oscillator is concerned, $t+2\pi=t$, which is of course the reason oscillators are used as clocks. This behaviour of the oscillator is reflected in the Lie group of its equation of motion by the existence of the compact subgroup SO(3) containing the time translation operation. What is perhaps more surprising is the fact that the existence of periodic motion is reflected in the existence of a compact subalgebra, that is at the local level. The direct physical question settled by the global discussion of this section was not whether the motion is periodic, but rather whether the motion has period 2π or 4π!

6. The global Lie group of the oscillator

Because the operators $X_1 \ldots X_8$ are of Lie's type and close under commutation, it follows from the converse of his third theorem that they integrate to an eight-parameter global Lie group. It is well known that the algebra A_2 can only generate three Lie groups: SU(3), SU(2, 1), and SL(3, R). As only the last of these is both non-compact and in possession of an SO(3) subgroup, the results of the previous sections identify SL(3, R) as the global Lie group of Newton's or Lagrange's equation for the oscillator.

7. Transformation of solutions

Any transformation that leaves differential equation (1) invariant necessarily transforms a solution $x=f(t)$ into a solution $x'=f'(t')$. If a point transformation is viewed in the passive sense, it is viewed as simply re-expressing a solution $f(t)$ in a new coordinate system as $f'(t')$. The problem of deciding whether $f'(t')$ is to be considered to represent the same or a different state of a physical system is the problem of deciding which observers are to be considered equivalent. This is a question that requires a consideration of a wide variety of physical systems as part of its resolution. It is not our purpose to enter into such considerations here—we shall not consider the question whether the active and passive view of the transformations of our SL(3, R) group are equivalent. Interesting physical information accrues when the transformations are viewed in the active sense. Then one may distinguish between transformations that change the functional dependence of a solution upon the independent variable, and transformations whose effect is to change the value of the dependent and independent variables while leaving the functional form of a solution unchanged. This analysis implies a fixed reference point in time.

An extended point transformation

$$(x, t, \dot{x}) \rightarrow (x', t', \dot{x}') \tag{36}$$

in general carries

$$x=f(t) \quad \text{into} \quad x'=f'(t')$$

and

$$g(t) \quad \text{into} \quad \dot{x}'=g'(t').$$

Only when

$$f'(t') = f(t'), \qquad \text{and hence} \qquad g'(t') = g(t'),$$

is the functional form of the solution unchanged.

For the equation of motion of the oscillator the fundamental existence theorem of ordinary differential equations allows one to suppose that at each point t for which a solution exists there is one and only one solution with given values of x and $\dot{x}$. Let us therefore specify the functional form of a solution by specifying the value of x and $\dot{x}$ that the solution takes on at some time t_0. If

$$f(t_0) = A, \qquad g(t_0) = B \tag{37}$$

we shall write

$$x = f_{AB}(t). \tag{38}$$

Now let

$$\hat{X} = \xi(x, t)\partial/\partial x + \eta(x, t)\partial/\partial t + \xi'(x, t, \dot{x})\partial/\partial\dot{x} \tag{39}$$

be the generator of a one-parameter group $T(a)$ that leaves the initial value A invariant, i.e., a group such that, if

$$(x, t, \dot{x}) = (f(t), t, g(t)) \to (A, t_0, B) \qquad \text{as} \qquad t \to t_0 \tag{40}$$

then

$$(x', t', \dot{x}') = (f'(t'), t', g'(t')) \to (A, t_0, B') \qquad \text{as} \qquad t \to t_0. \tag{41}$$

Now for an infinitesimal transformation $T(\delta a)$ we must have

$$(x', t', \dot{x}') = (x, t, \dot{x}) + \delta a(\xi, \eta, \xi') = (f(t), t, \dot{x}) + \delta a(\xi, \eta, \xi') \tag{42}$$

and this is to be an identity in δa. Because we need only consider terms to first order in δa it is a matter of indifference whether we suppose ξ, η, ξ' to be functions of the transformed or the untransformed variables. As we must have

$$\xi, \eta \to 0 \qquad \text{as} \qquad t \to t_0 \tag{43}$$

we may conclude that if the initial value A is to be invariant

$$\xi(A, t_0) = 0, \qquad \eta(A, t_0) = 0. \tag{44}$$

An exactly parallel argument shows that in order to have a one-parameter subgroup which leaves the slope B invariant, its generator must be such that

$$\eta(A, t_0) = 0, \qquad \xi'(A, t_0, B) = 0. \tag{45}$$

If we let $t_0 = 0$, then the general solution of the equation of motion of the oscillator is of course

$$f_{AB} = x = A\cos t + B\sin t. \tag{46}$$

Applying the results of the previous paragraphs we see that

$$\hat{X}_1 + \hat{X}_2, \qquad \hat{X}_5 - \hat{X}_8, \qquad \hat{X}_6 - \hat{X}_7 \tag{47}$$

are generators of one-parameter groups that leave the initial amplitude A invariant, while

$$\hat{X}_3 + \hat{X}_4, \qquad \hat{X}_6 + \hat{X}_7 \tag{48}$$

are generators of one-parameter groups that leave the initial velocity B invariant. If one allows linear combinations of generators with coefficients that are functions of A, B, these two sets may be enlarged.

In using these results it is important to distinguish a point transformation and a functional transformation. Thus the transformation from

$$x = f_{AB}(t) = A \cos t + B \sin t \tag{49}$$

$$x = A \cos(t+\tau) + B \sin(t+\tau) \tag{50}$$

that is carried out by the operator $T_5(\tau) = \exp \tau X_5$, does not in general yield the state of the oscillator that $f_{AB}(t)$ evolves into after a time τ. Such a time-evolved state would in general be developed by a point transformation whose generator changes x into $x' \neq x$, as well as t into t'. Only in the exceptional case $f_{AB}(t) = f_{AB}(t+\tau)$, (central in § 5), does T_5 act as a time evolution operator.

A point transformation

$$(x, t) \to (x', t') = (T(a)x, T(a)t) = (\Phi(x, t, a), \Psi(x, t, a)) \tag{51}$$

induces the functional transformation

$$x = f(t) \to x' = f'(t') = \Phi(T(a)^{-1}x', T(a)^{-1}t', a). \tag{52}$$

The infinitesimal version of this gives

$$f'(t') = f(t') + \delta a[\xi(f(t'), t') - g(t') \cdot \eta(f(t'), t')] \tag{53}$$

where, as before, g is the derivative of f, and $t' = t + \delta a \cdot \eta$. Thus for $f'(t')$ to be simply $f(t')$, as one must have for a general solution $x = f(t)$ to be transformed into $x' = f(t')$, it must be true that

$$\xi = \dot{x}\eta. \tag{54}$$

As a point transformation in x, t space cannot depend upon $\dot{x}$, it follows that there is no single operator in our realization of SL(3, R) that converts every solution $x = f(t)$ into $x' = f(t')$. However particular solutions may be so changed by particular operators.

Although there is no operator that carries a general solution into itself at an earlier or later time as one varies the group parameters, there are operators that transform the manifold of solutions $x = x(t)$ into itself without changing time intervals, i.e. such that

$$\mathrm{d}^2x'/\mathrm{d}t^2 + x' = 0 \qquad \text{if} \qquad \mathrm{d}^2x/\mathrm{d}t^2 + x = 0. \tag{55}$$

The necessary and sufficient condition for this to be true independently of a is that the generator of the transformation be such that

$$\mathrm{d}^2\xi/\mathrm{d}t^2 + \xi = 0 \qquad \text{on} \qquad x = x(t). \tag{56}$$

An alternative viewpoint is also revealing. Noting that x' may be expressed as

$$x' = \mathrm{e}^{aX''}x \tag{57}$$

it follows that on the solution surface $x = x(t)$

$$[\mathrm{e}^{aX''}(\mathrm{d}^2/\mathrm{d}t^2 + 1) - (\mathrm{d}^2/\mathrm{d}t^2 + 1)\,\mathrm{e}^{aX''}]x = 0. \tag{58}$$

For this to be true for all values of the parameter requires that

$$[X'', \mathrm{d}^2/\mathrm{d}t^2]x = 0. \tag{59}$$

In evaluating this commutator it is necessary to keep in mind that wherever x and its derivatives appear they are implicit functions of t. The transformations that satisfy (56) or (59) are found to be those generated by X_1+X_2, X_3+X_4, and X_5. They convert the solutions $x=f_{AB}(t)$ into $x'=f_{A'B'}(t)$.

All other transformations of the group convert the general solution

$$x=f_{AB}(t)=A\cos t+B\sin t \tag{60}$$

into

$$x'=f_{A'B'}(t')=A'\cos t'+B'\sin t', \qquad t'\neq t. \tag{61}$$

The dependence of these solutions upon t may be quite complicated, as may be seen, for example, by substituting $x(t)$ into equation (24). However, from an inspection of equations (14) it will be seen that every group generator that is t dependent depends upon t only via $\sin t$ or $\cos t$. *Consequently every function $f_{A'B'}(t'(x(t), t; a))$ obtainable from a solution $x(t)$ by action of the oscillator group $SL(3, R)$ is periodic in both t and t' with period 2π.*

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